Bell's Inequality Violation on MATLAB

Somi Yun, Fakhar Zaman, Junaid ur Rehman, and Hyundong Shin
Department of Electronic Engineering, Kyung Hee University, Korea
Email: hshin@khu.ac.kr

***Abstract*—Bell's inequality provides a framework to objectively decide the locality of a system. In this article, we demonstrate the violation of Bell's inequality on IBM quantum devices and Matlab. Our main objective in Matlab simulations is to characterize the robustness of Bell's inequality under different types of noise models. We find parameter thresholds for depolarizing and amplitude damping channels below which violation of Bell's inequality can be ensured for the maximally entangled states.**

I. Introduction

Quantum mechanics and theory of relativity are two important pillars of modern physics. At the early stages of development of quantum mechanics, many researchers argued that the quantum mechanics is either incomplete (phenomenon of hidden variables) or incorrect. Einstein and his colleagues (famously called EPR) argued for the incompleteness of quantum mechanics based on the principle of locality. On the other hand, Bohr defended the integrity of quantum mechanics by stating nonlocality as a feature of quantum mechanics [1]. According to locality, two particles that are spatially distant cannot directly affect each other. In order for one particle to affect another, the space between the two must be mediated. Conversely, in nonlocality, two distant particles can construct one quantum state and are influenced by actions on each other.

Although both sides came up with plausible arguments, but EPR and Bohr's debate unfolded only through thought experiments and logic did not escape the realm of abstract philosophical debate. In the real world, there is no way to examine thought experiments. Later, Bell proposed physical experiments to check the validity of thought experiments being argued. Bell hypothesized that Einstein's theory of hidden variables was correct under local realism with locality as the basic premise. He proposed an inequality which must be satisfied for the local systems but could be easily violated for nonlocal systems [2], [3]. The theory of local hidden variables—*each object has unique attributes, but we simply cannot explain them because we do not know the hidden variables* which is one way of interpreting quantum mechanics. The knowledge of hidden variables makes it possible to predict physical quantities as in classical physics.

In quantum mechanics, Bell's inequality is violated due to two landmark features of quantum physics— nonlocality and coherence. Quantum coherence is based on the idea that particles such as electrons are described by a wave function [4]. If wave-like property of a particle is split in two, then the two waves may coherently interfere with each other in such a way as to form a correlated composite state. Among the quantum states, special states having coherence and nonlocal characteristics are called entangled states which are the basis of quantum mechanics.

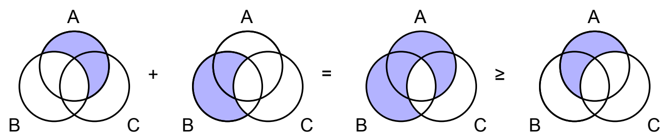


Fig. 1: A visual proof of Bell's inequality : $P\left(\mathrm{A}, \sim \mathrm{B}\right) + P\left(\mathrm{B}, \sim \mathrm{C}\right) \geq P\left(\mathrm{A}, \sim \mathrm{C}\right)$ using Venn diagram.

In this paper, we demonstrate the violation of Bell's inequality on a simulator created through MATLAB App Designer which models the gate-level noise of IBM quantum device with depolarizing and amplitude damping channels. We adjust modelling parameters p and η which represent the levels of quantum noise and find the limits of each parameters for Bell's inequality violated. We also show violation of Bell's inequality on IBM quantum device and validate the results of simulator.

II. Violation of Bell's inequality

The Bell's inequality experimentally describes the theory of local hidden variables and quantum mechanics as follows:

$$P\left(\mathrm{A}, \sim \mathrm{B}\right) + P\left(\mathrm{B}, \sim \mathrm{C}\right) \geq P\left(\mathrm{A}, \sim \mathrm{C}\right) \tag{1}$$

Fig. 1 shows the Venn diagram of Bell's inequality. However, the quantum entangled states can not be explained by the classical structure that follows the principle of locality [5], [6]. The singlet state $|\phi\rangle$ is the most representative example of violating Bell's inequality where $|\phi\rangle$ is defined as follows:

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{2}$$

From Fig. 2, A is a spin up event along z-axis, B is the event of spin up along 45° from z-axis and C is the event of spin up along x-axis. Thus $P\left(\mathrm{A}, \sim \mathrm{B}\right)$ is probability of particle-1 in event A and particle-1 is not in event B, $P\left(\mathrm{B}, \sim \mathrm{C}\right)$ is probability of particle-1 in event B and particle-1 is not in event C and $P\left(\mathrm{A}, \sim \mathrm{C}\right)$ is probability of particle-1 in event A and particle-1 is not in event C [7], respectively. Since the initial state is singlet the probability that particle-1 is in event B (resp. C) is same as particle-2 is not in event B (resp. C).

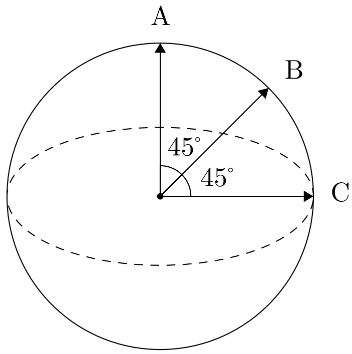


Fig. 2: Spin directions of event A, B and C .

To demonstrate the violation of Bell's inequality, we apply measurements (projection operators) to calculate the probabilities as follows

$$\begin{aligned} P\left(\mathrm{A}, \sim \mathrm{B}\right) &= \left\langle \phi \right| \Pi_{\mathrm{A}} \otimes \Pi_{\mathrm{B}} \left| \phi \right\rangle \\ &= \frac{1}{4} - \frac{1}{4\sqrt{2}} \end{aligned} \tag{3}$$

$$\begin{aligned} P\left(\mathrm{B}, \sim \mathrm{C}\right) &= \left\langle \phi \right| \Pi_{\mathrm{B}} \otimes \Pi_{\mathrm{C}} \left| \phi \right\rangle \\ &= \frac{1}{4} - \frac{1}{4\sqrt{2}} \end{aligned} \tag{4}$$

$$\begin{aligned} P\left(\mathrm{A}, \sim \mathrm{C}\right) &= \left\langle \phi \right| \Pi_{\mathrm{A}} \otimes \Pi_{\mathrm{C}} \left| \phi \right\rangle \\ &= \frac{1}{2}, \end{aligned} \tag{5}$$

where

$$\Pi_{\mathrm{A}} = \frac{1}{2}\left(1 + Z\right) \tag{6}$$

$$\Pi_{\mathrm{B}} = \frac{1}{2}\Big(1 + \frac{1}{\sqrt{2}}\left(X + Z\right)\Big) \tag{7}$$

$$\Pi_{\mathrm{C}} = \frac{1}{2}\left(1 + X\right), \tag{8}$$

and X (Z) denotes single qubit Pauli x (z) operator. From equation (3)-(5)

$$\begin{aligned} P\left(\mathrm{A}, \sim \mathrm{B}\right) + P\left(\mathrm{B}, \sim \mathrm{C}\right) &= \frac{1}{2} - \frac{1}{2\sqrt{2}} \\ &\ngeq P\left(\mathrm{A}, \sim \mathrm{C}\right). \end{aligned} \tag{9}$$

Therefore, it violates the Bell's inequality.

III. Bell's inequality Violation on MATLAB

Fig. 3 shows the schematic of MATLAB App Designer to demonstrate the violation of Bell's inequality. Initially, we prepare the singlet state by using ideal gates. We use R_z gates to determine the spin direction particle-1 and particle-2 followed by performing the Hadamard gate H and performing measurement in computational basis where R_z is the rotation matrix around z-axis. In practical, either the Bell's inequality is violated or not depends on the noise level in quantum gates. In order to adjust gate error rates, we use MATLAB App Designer to simulate the non ideal gates and create circuits.

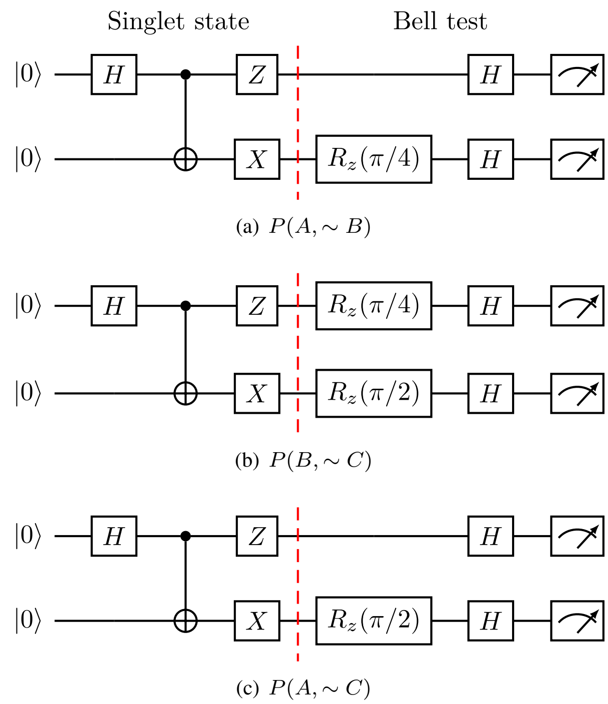


Fig. 3: Circuits of violation of Bell's inequality that can be divided into two parts: preparation of the singlet state as the initial state and the Bell test. (a) $P(A, \sim B)$ is the probability of finding 00 in the circuit (a) measurement result, (b) $P(B, \sim C)$ is the probability of finding 00 in the circuit (b) measurement result, (c) $P(A, \sim C)$ is the probability of finding 00 in the circuit (c) measurement result.

A. Gate Models

Quantum noise can be characterized by different quantum channels, e.g., depolarizing and amplitude damping channel [8]. Depolarizing channel maps a state ρ onto a convex combination of ρ and the maximally mixed state:

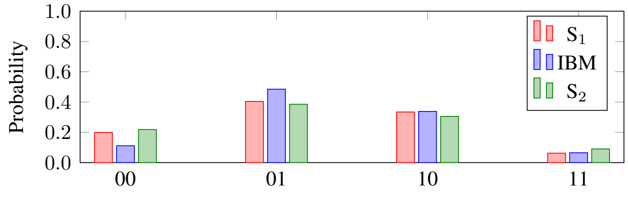
$$\mathcal{N}_{\mathrm{D}}\left(\rho\right) = \left(1 - p\right)\rho + p\pi, \tag{10}$$

where p is the depolarizing noise parameter and π denotes the maximally mixed state, respectively. However, the results of IBM quantum implementation suggest the noise of nonunital nature. Therefore, we additionally apply amplitude damping channel to characterize errors of non ideal gates.

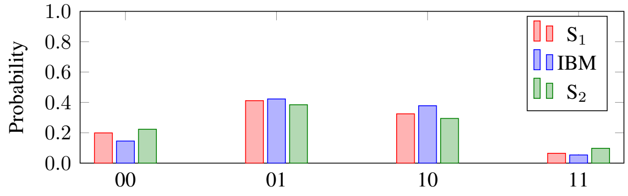
The amplitude damping channel is a channel which produces the asymmetric decay of the excited state $|1\rangle$ to the ground state $|0\rangle$ with probability η. The action of the channel on the two-qubit input state ρ is

$$\mathcal{N}_{\mathrm{A}}\left(\rho\right) = \sum_{j=0}^{1}\sum_{i=0}^{1}\left(K_i \otimes K_j\right)\rho\left(K_i \otimes K_j\right)^{\dagger}, \tag{11}$$

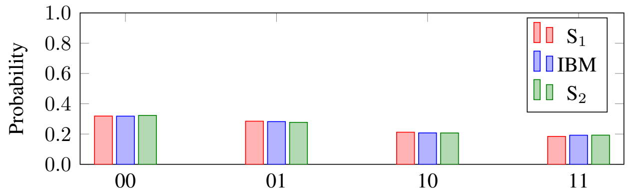
where η is the damping parameter and the Kraus operators K_0 and K_1 for the one-qubit amplitude damping channel are



(a) $P(A, \sim B)$



(b) $P(B, \sim C)$



(c) $P(A, \sim C)$

Fig. 4: Measurement statistics from MATLAB App Designer and IBM quantum device. S_1 is the simulation with the noiseless singlet state and the noisy Bell test. S_2 is the simulation with the noisy singlet state and the noisy Bell test which works similar to IBM.

defined as

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & \sqrt{\eta} \\ 0 & 0 \end{bmatrix}. \tag{12}$$

We model nonideal Pauli-Z, Pauli-X, CNOT and R_z by applying depolarizing channel to an ideal gate. By comparing the results of IBM quantum device with the theoretical result, we get modeling parameter p. However, measurement results applying a non ideal Hadamard gate to a state $|0\rangle$ is independent of p. To model nonideal Hadamard gates, we apply the amplitude damping channel before depolarizing channel and get modeling parameter p and η. Table I shows modeling parameters of gates mentioned.

In Fig. 4, we plot simulation of Bell's inequality violation using noiseless singlet state on MATLAB App Designer (pink bars), IBM quantum devices (blue bars) and noisy singlet state on MATLAB App Designer (green bars).

B. Limits of modeling parameter of each gates

As shown in Fig. 3, our schematic is divided into state preparation and Bell test circuit. We assume the ideal gates for state preparation to generate noiseless initial states. In the Bell test, we have three noise parameters (p_1, p_2, η) where p_1 and η are the noise parameters of Hadamard gates and p_2 denotes the noise level in R_z gates, respectively. We find the tetrahedron region of Bell's inequality violation (see Fig. 5) where the four corners of the tetrahedron are (0,0,0), (0.1591,0,0), (0,0.21,0) and (0,0,0.03414), respectively. Fig. 5 shows the parameter space for the noise models we use. The tetrahedron shown here is an approximate depiction for the parameters' range for the violation of Bell's inequality. Violation of Bell's inequality can be ensured within the shown tetrahedron with the maximal violation at the origin.

TABLE I: Gate errors

Nonideal gate	Modeling parameter
Pauli-Z	p = 0.008
Pauli-X	p = 0.162
CNOT	p = 0.0167
R_z	p = 0.02172
H	p = 0.00244, η = 0.05776

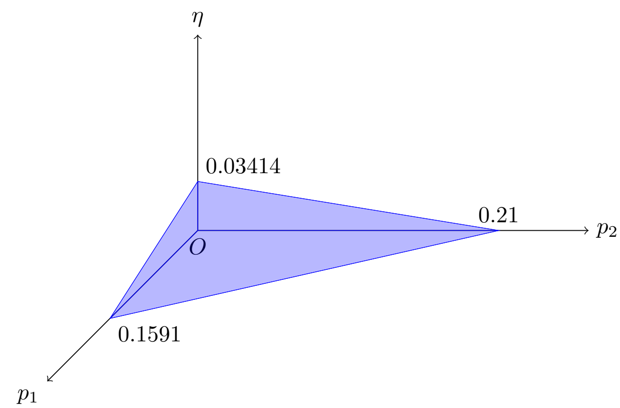


Fig. 5: The tetrahedron region of Bell's inequality violation.

IV. Validation on IBM quantum device.

IBM provides remote access to 5-qubit and 16-qubit quantum computers located in Melbourne, Ourense, Vigo and Yorktown-bmqx2 for quantum computing. We use Qiskit which is an open source framework based on Python to create, compile and run quantum computing programs. To validate our results, we implemented the circuit shown in Fig. 3 on IBM quantum (IBMQ) devices. The measurement results on IBM quantum device found for $P(\mathrm{A}, \sim \mathrm{B})$, $P(\mathrm{B}, \sim \mathrm{C})$ and $P(\mathrm{A}, \sim \mathrm{C})$ are obtained using 5-qubit quantum computer and compared with simulation results on MATLAB App Designer (see Fig. 4). In practical scenarios, it is impossible to prepare noiseless initial states. To make simulation results of MATLAB App Designer comparable with IBMQ results, we introduced the error parameters (p_1, p_2, η) for both stages state preparation and Bell test on MATLAB App Designer. In Fig. 4, we plot simulation of Bell's inequality violation using noiseless singlet state on MATLAB App Designer (pink bars), IBMQ devices (blue bars) and noisy singlet state on MATLAB App Designer (green bars).

V. Conclusion

We simulated the violation of Bell's inequality on a simulator designed on the MATLAB App Designer. The simulator is capable of simulating noiseless and noisy quantum circuits. We verify the simulator results by comparing them with the results obtained on IBM quantum device. We implemented the circuits which have the noiseless initial state and the noisy Bell test part to demonstrate the Bell's inequality violation and prove the quantum nonlocality using quantum entanglement. Future works may include design of simulators that incorporate more noise models. Another possible approach is to adapt a hybrid approach for the simulation of quantum circuits where some part of the circuit is simulated on classical computer and other on IBM device.

Acknowledgments

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